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# Summarizing Measured Data

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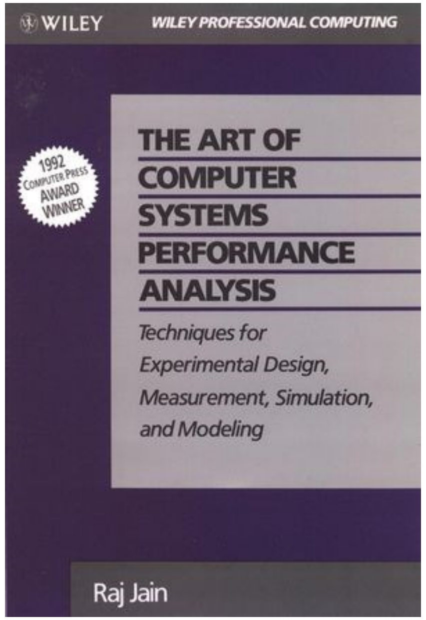
# References

- Raj Jain, **The Art of Computer Systems Performance Analysis**, Wiley, 1991.
  - Part I: An Overview of Performance Evaluation
  - Part II: Measurement Techniques and Tools
  - Part III: Probability Theory and Statistics
  - Part IV: Experimental Design and Analysis
  - Part V: Simulation



# Outline

- Summarizing Data by a Single Number
  - Mean, Median, and Mode
  - Common Misuses of Means
  - Geometric Mean
  - Harmonic Mean
- Mean of a Ratio
- Summarizing Variability
  - Range
  - Variance
  - Percentiles
  - Semi Inter-Quartile Range
  - Mean Absolute Deviation
  - Selecting the Index of Dispersion

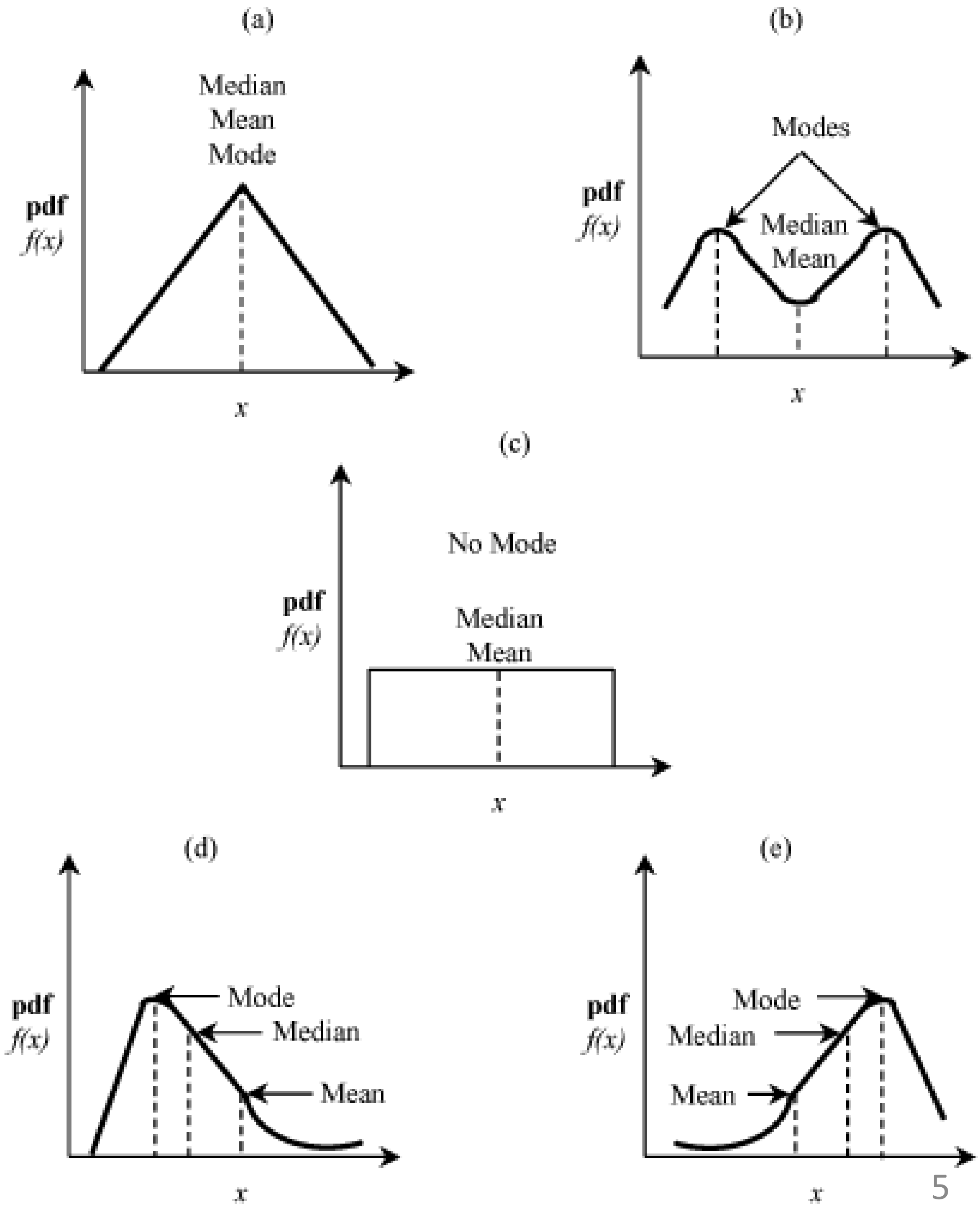
# Mean, Median, and Mode

Are called **indices of central tendencies**.

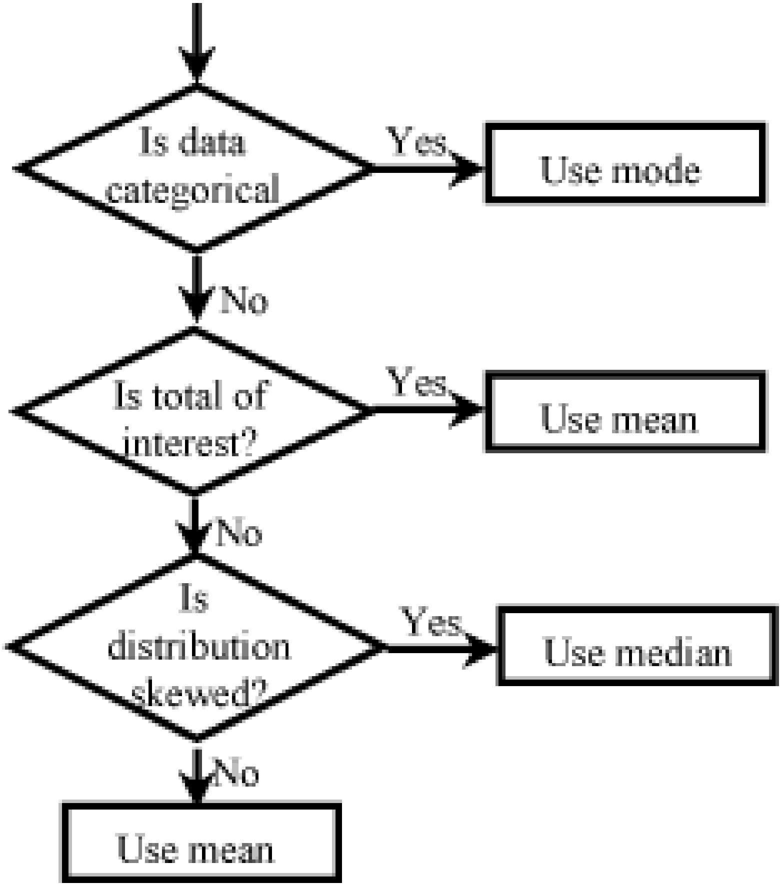
- **Mean:** $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$
- **Median** is obtained by sorting the observations and taking the observation that is in the middle of the series.
- **Mode** is obtained by plotting a histogram and specifying the midpoint of the bucket where the histogram peaks. For **categorical** variables, mode is given by the category that occurs most frequently.

# Relationships

- **Mean** and **median** always exist and are unique.
- **Mode**, on the other hand, may not exist or may not be unique.



# Selecting Mean, Median, and Mode



# Examples

1. **Used resources in a system**
   - Resources are categorical and hence **mode** must be used.
2. **Interarrival time of service requests**
   - Total time is of interest and so **mean** is the proper choice.
3. **Load on a computer**
   - **Median** is preferable due to a highly skewed distribution.
4. **Average configuration of many computers**
   - **Medians** of number devices, memory sizes, number of processors are generally used to specify the configuration due to the skewness of the distribution.

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# Common Misuses of Means

1. Using mean of **significantly different values**
    - (10+1000)/2 = 505

2. Using mean without regard to the **skewness of distribution**.

| System A | System B |
|---:|---:|
| 10 | 5 |
| 9 | 5 |
| 11 | 5 |
| 10 | 4 |
| 10 | 31 |
| Sum=50 | Sum=50 |
| Mean=10 | Mean=10 |
| Typical=10 | Typical=5 |

# Common Misuses of Means (cont.)

3. **Multiplying means to get the mean of a product**

$$E(xy) \neq E(x)E(y)$$

| | | | |
|---|---|---|---|
| | *A* | *B* | *AB* |
| | 2 | 5 | 10 |
| | 3 | 6 | 18 |
| | 4 | 7 | 28 |
| Average | **3** | **6** | **18.7** |

4. **Taking a mean of a ratio with different bases**
   - Already discussed in ratio games

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# Geometric Mean

- Is used if the **product of the observations is a quantity of interest**.

$$\dot{x} = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$$

- **Example**: The performance improvements in 7 layers:

| Protocol Layer | Performance Improvement |
|---|---|
| 7 | 18% |
| 6 | 13% |
| 5 | 11% |
| 4 | 8% |
| 3 | 10% |
| 2 | 28% |
| 1 | 5% |

Average improvement per layer

$$= \{(1.18)(1.13)(1.11)(1.08)(1.10)(1.28)(1.05)\}^{\frac{1}{7}} - 1$$

$$= 0.13$$

# Examples of Multiplicative Metrics

1. Cache hit ratios over several levels of caches
2. Percentage performance improvement between successive versions
3. Average error rate per hop on a multi-hop path in a network

# Geometric Mean of Ratios

- The **geometric mean of a ratio** is the **ratio** of the geometric means of the **numerator** and **denominator**.

$$gm\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_n}{y_n}\right) = \frac{gm(x_1, x_2, \cdots, x_n)}{gm(y_1, y_2, \ldots, y_n)} = \frac{1}{gm\left(\frac{y_1}{x_1}, \frac{y_2}{x_2}, \ldots, \frac{y_n}{x_n}\right)}$$

- The choice of the **base** in relative performance **does not change the conclusion** when comparing two systems.
- Therefore, the geometric mean is **recommended for relative performance**, *e.g.*, SPEC CPU benchmark.

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# Harmonic Mean

- Used whenever an **arithmetic mean can be justified for 1/$x_i$**

$$\ddot{x} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

- **Example**: MIPS of a benchmark on a processor
- In the *i*-th repetition, the benchmark takes $t_i$ seconds. Now suppose the benchmark has *m* million instructions, MIPS $x_i$ computed from the *i*-th repetition is:

$$x_i = \frac{m}{t_i}$$

- $t_i$'s should be summarized using arithmetic mean since the sum of $t_i$ has a physical meaning => $x_i$'s should be summarized using harmonic mean since the sum of 1/$x_i$'s has a physical meaning.

# Outline

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  - Four cases
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# Mean of a Ratio

1. If the **sum of numerators** and the **sum of denominators**, both **have physical meanings**, the average of the ratio is the **ratio of the averages**.
   - For $x_i = a_i / b_i$, the average ratio is given by:

$$\begin{aligned}\text{Average}\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots, \frac{a_n}{b_n}\right) &= \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \\ &= \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \\ &= \frac{\frac{1}{n}\sum_{i=1}^{n} a_i}{\frac{1}{n}\sum_{i=1}^{n} b_i} = \frac{\bar{a}}{\bar{b}}\end{aligned}$$

# Mean of a Ratio (cont.)

- **Example: CPU utilization**

- **Note**: Ratios cannot always be summarized by a geometric mean. A geometric mean of utilizations is useless.

| Measurement Duration | CPU Busy |
|---|---|
| 1 | 45% |
| 1 | 45% |
| 1 | 45% |
| 1 | 45% |
| 100 | 20% |
| Sum | 200 |
| Mean | $\neq 200/5$ or 40% |

$$
\begin{aligned}
\text{Mean CPU utilization} &= \frac{\text{Sum of CPU busy times}}{\text{Sum of measurement durations}} \\
&= \frac{0.45 + 0.45 + 0.45 + 0.45 + 20}{1 + 1 + 1 + 1 + 100} \\
&= 21\%
\end{aligned}
$$

# Mean of a Ratio (cont.)

2. If the **denominator is a constant** and the **sum of numerator** has a **physical meaning**, the **arithmetic mean** of the ratios can be used.

That is, if $b_i = b$ for all $i$'s, then:

$$\text{Average}(\frac{a_1}{b}, \frac{a_2}{b}, \cdots, \frac{a_n}{b})$$
$$= \frac{1}{n}\left(\frac{a_1}{b} + \frac{a_2}{b} + \cdots + \frac{a_n}{b}\right)$$
$$= \frac{\sum_{i=1}^{n} a_i}{nb}$$

**Example**: Mean resource utilization over same period

# Mean of a Ratio (cont.)

3. If the **sum of the denominators** has a **physical meaning** and the **numerators are constant**, then a **harmonic mean** of the ratio should be used to summarize them.

That is, if $a_i = a$ for all $i$'s, then:

$$\text{Average}\left(\frac{a}{b_1}, \frac{a}{b_2}, \cdots, \frac{a}{b_n}\right) = \frac{n}{\frac{b_1}{a} + \frac{b_2}{a} + \cdots + \frac{b_n}{a}}$$
$$= \frac{na}{\sum_{i=1}^{n} b_i}$$

**Example**: MIPS using the same benchmark (see Harmonic Mean)

# Mean of a Ratio (cont.)

4. If the **numerator and the denominator** are expected to **follow a multiplicative property** such that $a_i = c \times b_i$, where $c$ is approximately a constant that is being estimated, then $c$ can be estimated by the **geometric mean** of $a_i / b_i$.

**Example**: Program optimizer

Where, $b_i$ and $a_i$ are the sizes before and after the program optimization and $c$ is the effect of the optimization, which is expected to be independent of the code size.

| | Code Size | | |
|---|---|---|---|
| Program | Before | After | Ratio |
| BubbleP | 119 | 89 | 0.75 |
| IntmmP | 158 | 134 | 0.85 |
| PermP | 142 | 121 | 0.85 |
| PuzzleP | 8612 | 7579 | 0.88 |
| QueenP | 7133 | 7062 | 0.99 |
| QuickP | 184 | 112 | 0.61 |
| SieveP | 2908 | 2879 | 0.99 |
| TowersP | 433 | 307 | 0.71 |
| Geometric Mean | | | 0.79 |

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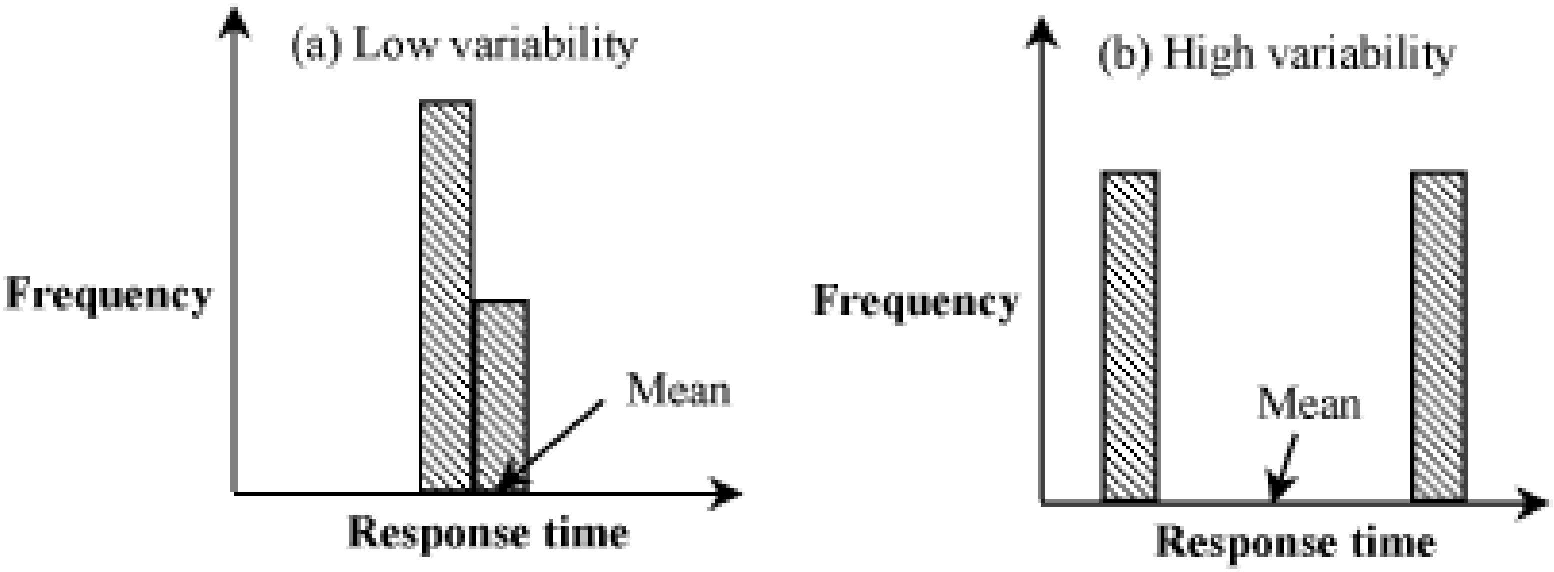
# Summarizing Variability

“*Then there is the man who drowned crossing a stream with an average depth of six inches*.”

- W. I. E. Gates



# Indices of Dispersion

1. Range
2. Variance
3. Percentiles
4. Semi inter-quartile range
5. Mean absolute deviation

# 1. Range

- **Range = Max - Min**
- Larger range => higher variability
- In most cases, range is **not very useful**; the minimum often comes out to be zero and the maximum comes out to be an "outlier" far from typical values.
- Unless the variable is bounded, the maximum goes on increasing with the number of observations and the minimum goes on decreasing.
- Range is **useful** if, and only if, the **variable is bounded**.

# 2. Variance

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$\text{where } \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

- The variance is expressed in units which are **square** of the units of the observations.
  => It is preferable to use **standard deviation** $s$.
- **Coefficient of variation** (**COV**) $= s/\bar{x}$ is even better because it takes the scale of measurement (unit of measurement) out of variability consideration.

# 3. Percentiles

- A $k$-th **percentile** is a score below which a given percentage $k$ of scores falls at or below which a given percentage falls.
- Specifying the **5-percentile** and the **95-percentile** of a variable has the same impact as specifying its **minimum** and **maximum**.
- It can be done for any variable, even for variables without bounds.

# 3. Percentiles (cont.)

- When expressed as a **fraction $\alpha$ between 0 and 1** (instead of a percent), the percentiles are also called **quantiles**.
  => 0.9-quantile is the same as 90-percentile.
- **Fractile** = quantile
- The percentiles at **multiples of 10%** are called **deciles**. Thus, the first decile is 10-percentile, the second decile is 20-percentile, and so on.

# 4. Semi Inter-Quartile Range

- **Quartiles** divide the data into **four parts** at 25%, 50%, and 75%.
  - 25% of the observations are ≤ the first quartile $Q_1$
  - 50% are ≤ the second quartile $Q_2$
  - 75% are ≤ the third quartile $Q_3$
- Notice that the second quartile $Q_2$ is also the median.
- **Inter-quartile range** = $Q_3 - Q_1$
- **Semi inter-quartile range** (**SIQR**)

$$\mathrm{SIQR} = \frac{Q_3 - Q_1}{2} = \frac{x_{0.75} - x_{0.25}}{2}$$

# How to Find a Quantile?

- The $\alpha$-quantiles can be estimated by **sorting** the observations and taking the $[(n\text{-}1)\alpha+1]$-th **element** in the ordered set. Here, [.] is used to denote rounding to the nearest integer.
- For quantities exactly half-way between two integers use the lower integer.

# Example

- In an experiment, which was repeated **32 times**, the measured CPU time was found to be {3.1, 4.2, 2.8, 5.1, 2.8, 4.4, 5.6, 3.9, 3.9, 2.7, 4.1, 3.6, 3.1, 4.5, 3.8, 2.9, 3.4, 3.3, 2.8, 4.5, 4.9, 5.3, 1.9, 3.7, 3.2, 4.1, 5.1, 3.2, 3.9, 4.8, 5.9, 4.2}.
- The **sorted set** is {1.9, 2.7, 2.8, 2.8, 2.8, 2.9, 3.1, 3.1, 3.2, 3.2, 3.3, 3.4, 3.6, 3.7, 3.8, 3.9, 3.9, 3.9, 4.1, 4.1, 4.2, 4.2, 4.4, 4.5, 4.5, 4.8, 4.9, 5.1, 5.1, 5.3, 5.6, 5.9}.
- **10-percentile** = [ 1+(31)(0.10) ] = 4th element = 2.8
- **90-percentile** = [ 1+(31)(0.90) ] = 29th element = 5.1
- **First quartile $Q_1$** = [ 1+(31)(0.25) ] = 9th element = 3.2
- **Median $Q_2$** = [ 1+(31)(0.50) ] = 16th element = 3.9
- **Third quartile $Q_3$** = [ 1+(31)(0.75) ] = 24th element = 4.5

$$\mathrm{SIQR} = \frac{Q_3 - Q_1}{2} = \frac{4.5 - 3.2}{2} = 0.65$$

# 5. Mean Absolute Deviation

$$\text{Mean absolute deviation} = \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$$

- **Fast** to calculate as no multiplication or square root is required.

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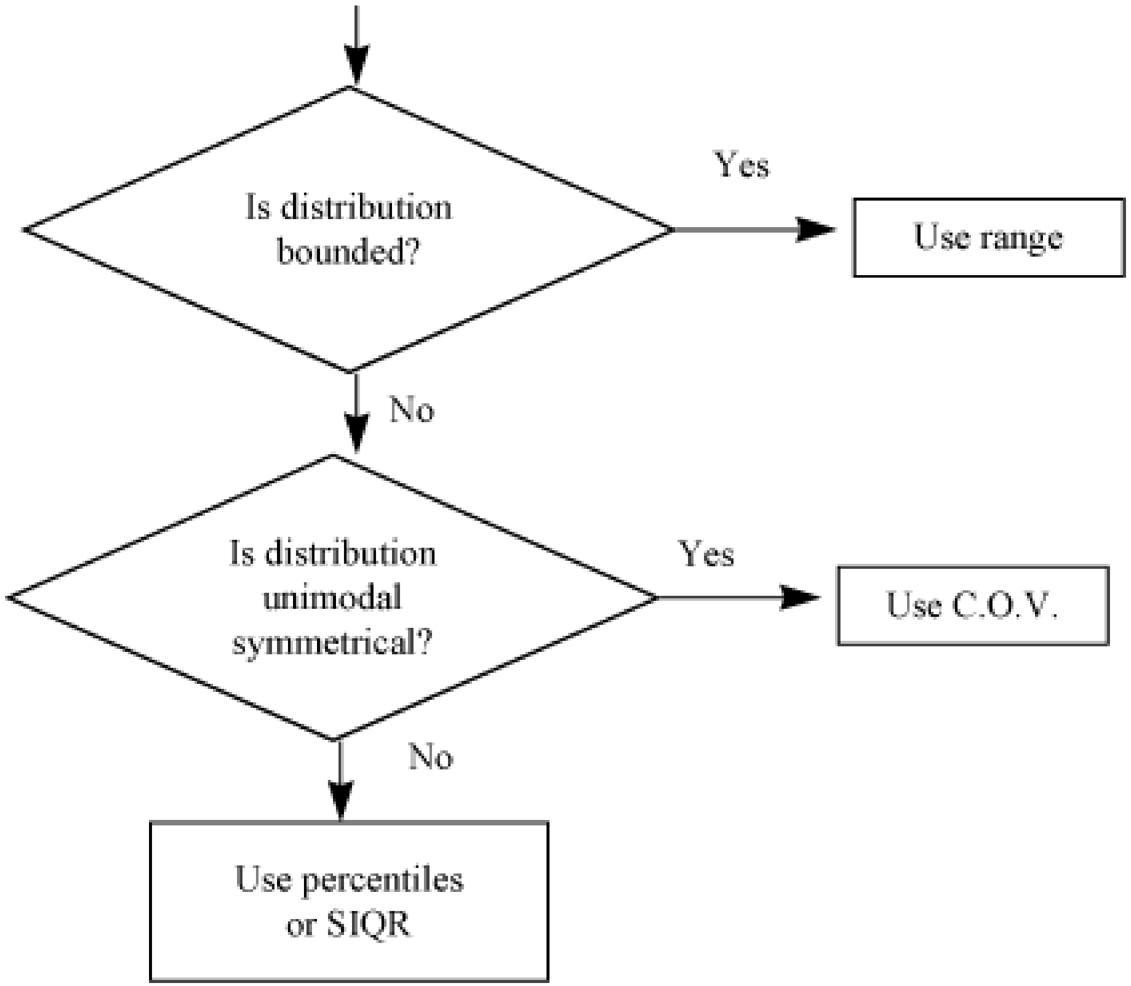

# Comparison of Variation Measures

- **Range** is affected considerably by outliers.
- **Variance** is also affected by outliers, but the affect is less.
- **Mean absolute deviation** is next in resistance to outliers.
- **Semi inter-quartile range** is very resistant to outliers.
- In general, **SIQR** is used as an index of dispersion whenever **median** is used.
- For **categorical data**, the dispersion can be specified by giving the **number of most frequent categories** that comprise the given percentile, *e.g.*, top 90%.

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