2^k Factorial Designs

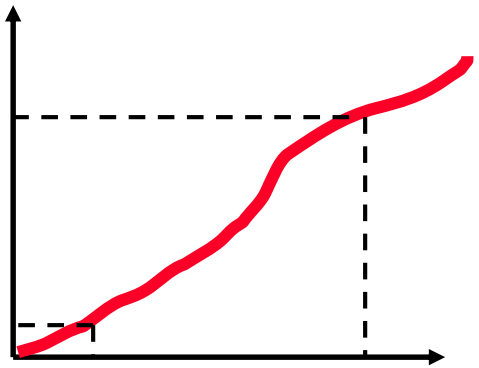
Adapted by Prof. Gheith Abandah

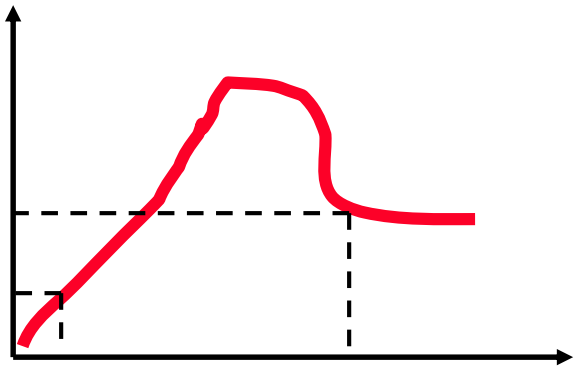
Overview

- 2^2 Factorial Designs
- Model
- Computation of Effects
- Sign Table Method
- Allocation of Variation
- General 2^k Factorial Designs

2^k Factorial Designs

- k factors, each at two levels.
- Easy to analyze.
- Helps in sorting out impact of factors.
- Good at the beginning of a study.
- Valid only if the effect is unidirectional.
  E.g., memory size, the number of disk drives





2^2 Factorial Designs

- Two factors, each at two levels.

Performance in MIPS

Cache Size	Memory Size	
	4M Bytes	16M Bytes
1K	15	45
2K	25	75

$$x_A = \begin{cases} -1 & \text{if 4M bytes memory} \\ 1 & \text{if 16M bytes memory} \end{cases}$$

$$x_B = \begin{cases} -1 & \text{if 1K bytes cache} \\ 1 & \text{if 2K bytes cache} \end{cases}$$

Model

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

Observations:

$$15 = q_0 - q_A - q_B + q_{AB}$$

$$45 = q_0 + q_A - q_B - q_{AB}$$

$$25 = q_0 - q_A + q_B - q_{AB}$$

$$75 = q_0 + q_A + q_B + q_{AB}$$

Solution:

$$y = 40 + 20x_A + 10x_B + 5x_A x_B$$

Interpretation: Mean performance = 40 MIPS

Effect of memory = 20 MIPS; Effect of cache = 10 MIPS

Interaction between memory and cache = 5 MIPS.

Computation of Effects

Experiment	A	B	y
1	-1	-1	y_1
2	1	-1	y_2
3	-1	1	y_3
4	1	1	y_4

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

$$y_1 = q_0 - q_A - q_B + q_{AB}$$

$$y_2 = q_0 + q_A - q_B - q_{AB}$$

$$y_3 = q_0 - q_A + q_B - q_{AB}$$

$$y_4 = q_0 + q_A + q_B + q_{AB}$$

Computation of Effects (Cont)

Solution:

$$q_0 = \frac{1}{4}(y_1 + y_2 + y_3 + y_4)$$

$$q_A = \frac{1}{4}(-y_1 + y_2 - y_3 + y_4)$$

$$q_B = \frac{1}{4}(-y_1 - y_2 + y_3 + y_4)$$

$$q_{AB} = \frac{1}{4}(y_1 - y_2 - y_3 + y_4)$$

Sign Table Method

I	A	B	AB	y
1	-1	-1	1	15
1	1	-1	-1	45
1	-1	1	-1	25
1	1	1	1	75
160	80	40	20	Total
40	20	10	5	Total/4

Allocation of Variation

- Importance of a factor = proportion of the *variation* explained

$$\text{Sample Variance of } y = s_y^2 = \frac{\sum_{i=1}^{2^2}(y_i - \bar{y})^2}{2^2 - 1}$$

$$\text{Total Variation of } y = \text{SST} = \sum_{i=1}^{2^2}(y_i - \bar{y})^2$$

- For a 2^2 design:

$$\text{SST} = 2^2 q_A^2 + 2^2 q_B^2 + 2^2 q_{AB}^2 = \text{SSA} + \text{SSB} + \text{SSAB}$$

- Variation due to A = SSA = $2^2 q_A^2$
- Variation due to B = SSB = $2^2 q_B^2$
- Variation due to interaction = SSAB = $2^2 q_{AB}^2$
- Fraction explained by A = $\frac{SSA}{SST}$ Variation ≠ Variance

Example 17.2

- Memory-cache study:

$$\bar{y} = \frac{1}{4}(15 + 55 + 25 + 75) = 40$$

$$\begin{aligned} \text{Total Variation} &= \sum_{i=1}^{4} (y_i - \bar{y})^2 \\ &= (25^2 + 15^2 + 15^2 + 35^2) \\ &= 2100 \\ &= 4 \times 20^2 + 4 \times 10^2 + 4 \times 5^2 \end{aligned}$$

- Total variation= 2100

  Variation due to Memory = 1600 (76%)

  Variation due to cache = 400 (19%)

  Variation due to interaction = 100 (5%)

General 2^k Factorial Designs

- k factors at two levels each.

  2^k experiments.

  2^k effects: k main effects

$$\left.\begin{pmatrix} k \\ 2 \end{pmatrix}\right\} \text{two factor interactions}$$

$$\left.\begin{pmatrix} k \\ 3 \end{pmatrix}\right\} \text{three factor interactions...}$$

2^k Design Example

- Three factors in designing a machine:
  - Cache size
  - Memory size
  - Number of processors

	Factor	Level -1	Level 1
A	Memory Size	4MB	16MB
B	Cache Size	1kB	2kB
C	Number of Processors	1	2

2^k Design Example (cont)

Cache Size	4M Bytes		16M Bytes	
	1 Proc	2 Proc	1 Proc	2 Proc
1K Byte	14	46	22	58
2K Byte	10	50	34	86

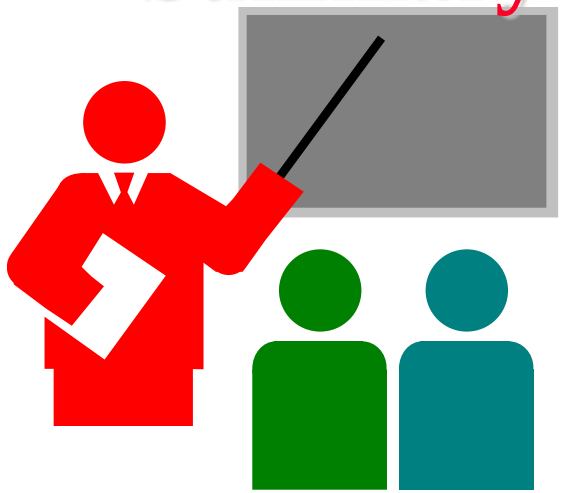
I	A	B	C	AB	AC	BC	ABC	y
1	-1	-1	-1	1	1	1	-1	14
1	1	-1	-1	-1	-1	1	1	22
1	-1	1	-1	-1	1	-1	1	10
1	1	1	-1	1	-1	-1	-1	34
1	-1	-1	1	1	-1	-1	1	46
1	1	-1	1	-1	1	-1	-1	58
1	-1	1	1	-1	-1	1	-1	50
1	1	1	1	1	1	1	1	86
320	80	40	160	40	16	24	9	Total
40	10	5	20	5	2	3	1	Total/8

Analysis of 2^k Design

$$\begin{aligned}
\text{SST} &= 2^3(q_A^2 + q_B^2 + q_C^2 + q_{AB}^2 + q_{AC}^2 + q_{BC}^2 + q_{ABC}^2) \\
&= 8(10^2 + 5^2 + 20^2 + 5^2 + 2^2 + 3^2 + 1^2) \\
&= 800 + 200 + 3200 + 200 + 32 + 72 + 8 = 4512 \\
&= 18\% + 4\% + 71\% + 4\% + 1\% + 2\% + 0\% \\
&= 100\%
\end{aligned}$$

- Number of Processors (C) is the most important factor.

Summary



- 2^k design allows k factors to be studied at two levels each
- Can compute main effects and all multi-factors interactions
- Easy computation using sign table method
- Easy allocation of variation using squares of effects