Comparing Systems Using Sample Data

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- Sample Versus Population
- Confidence Interval for The Mean
- Approximate Visual Test
- Sample Size for Determining Mean

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Sample Versus Population

- Generate several million random numbers with mean μ and standard deviation σ
  Draw a sample of *n* observations
- Sample mean ≠ population mean: $\bar{x} \neq \mu$
- Parameters: population characteristics
  = Unknown = Greek (μ and σ)
- Statistics: Sample estimates = Random = English ($\bar{x}$ and s)

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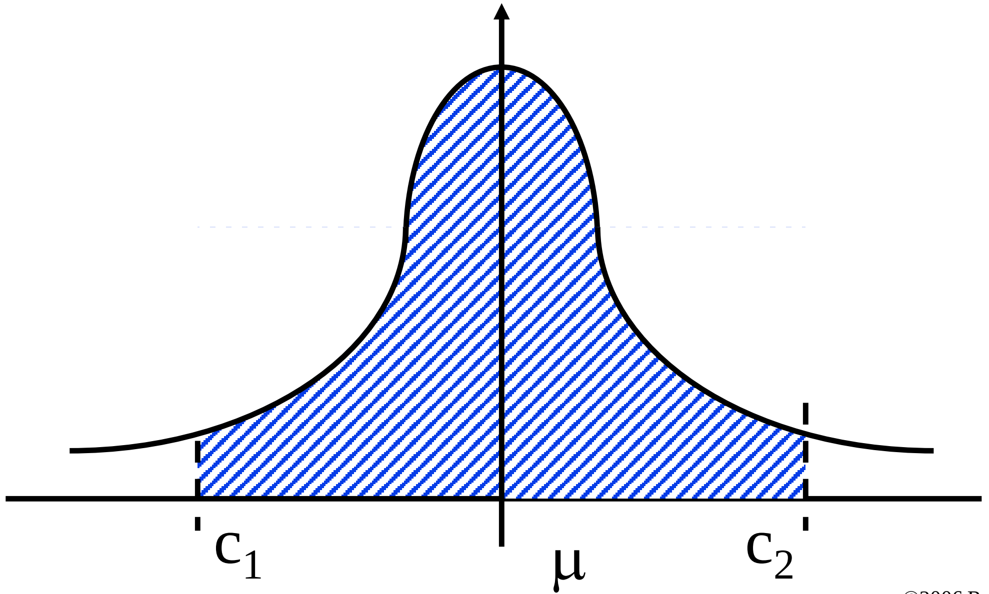
Confidence Interval for The Mean

- k samples $\Rightarrow$ k Sample means
  $\Rightarrow$ Can't get a single estimate of μ
  $\Rightarrow$ Use bounds c_1 and c_2:

$$\text{Probability}\{c_1 \leq \mu \leq c_2\} = 1-\alpha$$

- Confidence interval: $[(c_1, c_2)]$
- Significance level: α
- Confidence level: $100(1-\alpha)$
- Confidence coefficient: $1-\alpha$



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Determining Confidence Interval

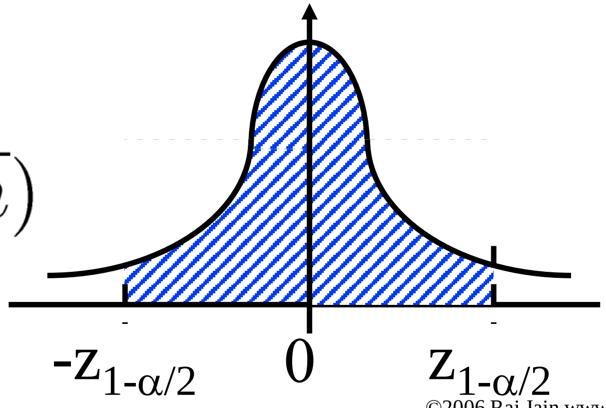
- Use 5-percentile and 95-percentile of the sample means to get 90% Confidence interval ⇒ Need many samples.
- Central limit theorem: Sample mean of independent and identically distributed observations:

$$\bar{x} \sim N(\mu, \sigma/\sqrt{n})$$

  Where μ = population mean, σ = population standard deviation
- Standard Error: Standard deviation of the sample mean $= \sigma/\sqrt{n}$
- 100(1-α)% confidence interval for μ:

$$(\bar{x} - z_{1-\alpha/2} s/\sqrt{n}, \bar{x} + z_{1-\alpha/2} s/\sqrt{n})$$

  $z_{1-\alpha/2}$ = (1-α/2)-quantile of N(0,1)



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A.2 QUANTITIES OF THE UNIT NORMAL DISTRIBUTION

Table A.2 lists z_p for a given p. For example, for a two-sided confidence interval at 95%, $\alpha = 0.05$ and $p = 1 - \alpha/2 = 0.975$. The entry in the row labeled 0.97 and column labeled 0.005 gives $z_p = 1.960$.

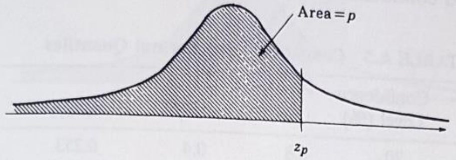

TABLE A.2 Quantiles of the Unit Normal Distribution

p	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.5	0.000	0.025	0.050	0.075	0.100	0.126	0.151	0.176	0.202	0.228
0.6	0.253	0.279	0.305	0.332	0.358	0.385	0.412	0.440	0.468	0.496
0.7	0.524	0.553	0.583	0.613	0.643	0.674	0.706	0.739	0.772	0.806
0.8	0.842	0.878	0.915	0.954	0.994	1.036	1.080	1.126	1.175	1.227

p	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.90	1.282	1.287	1.293	1.299	1.305	1.311	1.317	1.323	1.329	1.335
0.91	1.341	1.347	1.353	1.359	1.366	1.372	1.379	1.385	1.392	1.398
0.92	1.405	1.412	1.419	1.426	1.433	1.440	1.447	1.454	1.461	1.468
0.93	1.476	1.483	1.491	1.499	1.506	1.514	1.522	1.530	1.538	1.546
0.94	1.555	1.563	1.572	1.580	1.589	1.598	1.607	1.616	1.626	1.635
0.95	1.645	1.655	1.665	1.675	1.685	1.695	1.706	1.717	1.728	1.739
0.96	1.751	1.762	1.774	1.787	1.799	1.812	1.825	1.838	1.852	1.866
0.97	1.881	1.896	1.911	1.927	1.943	1.960	1.977	1.995	2.014	2.034
0.98	2.054	2.075	2.097	2.120	2.144	2.170	2.197	2.226	2.257	2.290

p	0.0000	0.0001	0.0002	0.0003	0.0004	0.0005	0.0006	0.0007	0.0008	0.0009
0.990	2.326	2.330	2.334	2.338	2.342	2.346	2.349	2.353	2.357	2.362
0.991	2.366	2.370	2.374	2.378	2.382	2.387	2.391	2.395	2.400	2.404
0.992	2.409	2.414	2.418	2.423	2.428	2.432	2.437	2.442	2.447	2.452
0.993	2.457	2.462	2.468	2.473	2.478	2.484	2.489	2.495	2.501	2.506
0.994	2.512	2.518	2.524	2.530	2.536	2.543	2.549	2.556	2.562	2.569
0.995	2.576	2.583	2.590	2.597	2.605	2.612	2.620	2.628	2.636	2.644
0.996	2.652	2.661	2.669	2.678	2.687	2.697	2.706	2.716	2.727	2.737
0.997	2.748	2.759	2.770	2.782	2.794	2.807	2.820	2.834	2.848	2.863
0.998	2.878	2.894	2.911	2.929	2.948	2.968	2.989	3.011	3.036	3.062
0.999	3.090	3.121	3.156	3.195	3.239	3.291	3.353	3.432	3.540	3.719

See Table A.3 for commonly used values.

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Example 13.1

- $\bar{x} = 3.90$, $s = 0.95$ and $n = 32$
- A 90% confidence interval for the mean
  $$= 3.90 \mp (1.645)(0.95)/\sqrt{32} = (3.62,\ 4.17)$$
- We can state with 90% confidence that the population mean is between 3.62 and 4.17 The chance of error in this statement is 10%.

$$\text{A 95\% confidence interval for the mean} = 3.90 \mp (1.960)(0.95)/\sqrt{32} = (3.57,\ 4.23)$$

$$\text{A 99\% confidence interval for the mean} = 3.90 \mp (2.576)(0.95)/\sqrt{32} = (3.46,\ 4.33)$$

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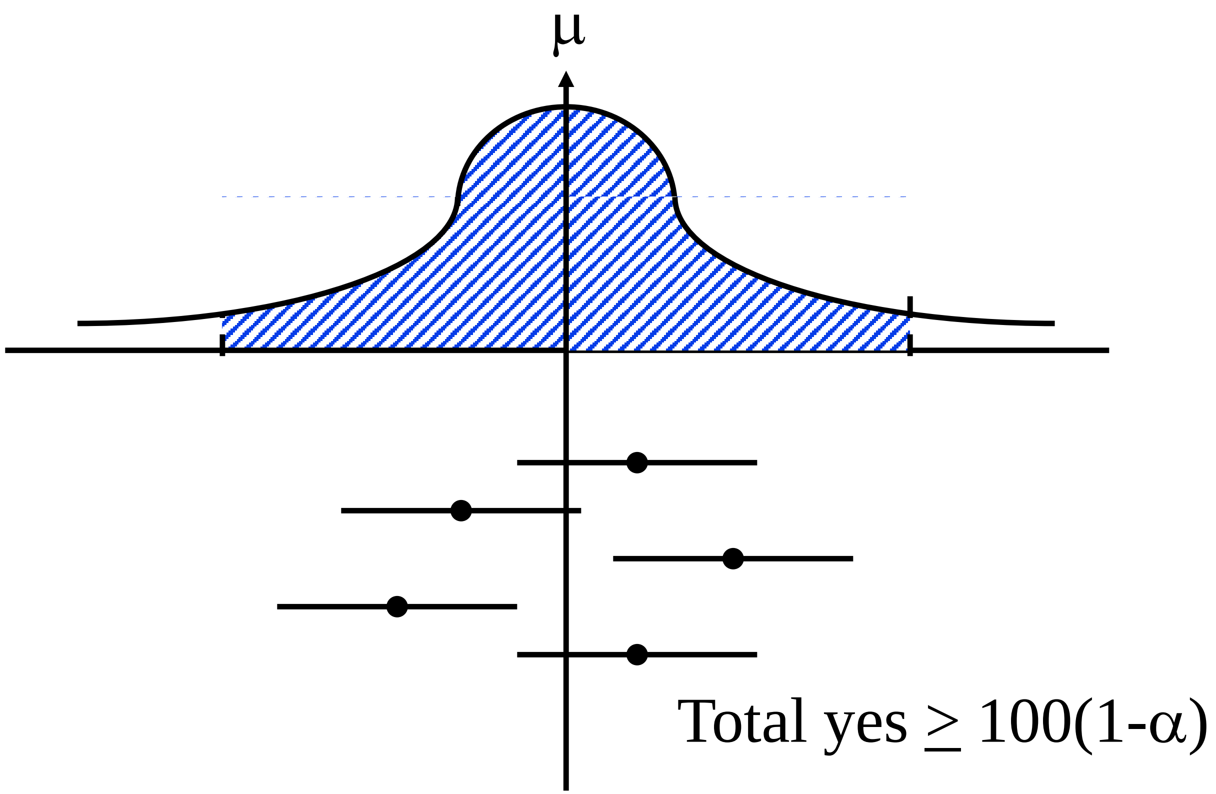
Confidence Interval: Meaning

- If we take 100 samples and construct confidence interval for each sample, the interval would include the population mean in 90 cases.



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Confidence Interval for Small Samples

- $100(1-\alpha)$ % confidence interval for $n < 30$:

$$(\bar{x} - t_{[1-\alpha/2;n-1]} s/\sqrt{n}, \bar{x} + t_{[1-\alpha/2;n-1]} s/\sqrt{n})$$

- $t_{[1-\alpha/2;\, n-1]}$ = $(1-\alpha/2)$-quantile of a t-variate with n-1 degrees of freedom

$$x \sim N(\mu, \sigma^2)$$
$$\Rightarrow (\bar{x} - \mu)/(\sigma/\sqrt{n}) \sim N(0,1)$$
$$(n-1)s^2/\sigma^2 \sim \chi^2(n-1)$$
$$(\bar{x} - \mu)/\sqrt{s^2/n} \sim t(n-1)$$

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A.4 QUANTILES OF THE *t* DISTRIBUTION

Table A.4 lists $t_{[p;n]}$. For example, the $t_{[0.95;13]}$ required for a two-sided 90% confidence interval of the mean of a sample of 14 observation is 1.771.

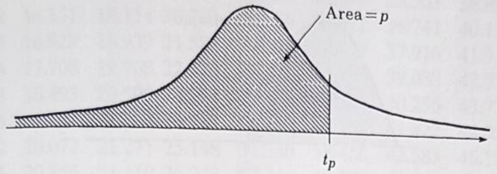

TABLE A.4 Quantiles of the *t* Distribution

	p							
n	0.6000	0.7000	0.8000	0.9000	0.9500	0.9750	0.9950	0.9995
1	0.325	0.727	1.377	3.078	6.314	12.706	63.657	636.619
2	0.289	0.617	1.061	1.886	2.920	4.303	9.925	31.599
3	0.277	0.584	0.978	1.638	2.353	3.182	5.841	12.924
4	0.271	0.569	0.941	1.533	2.132	2.776	4.604	8.610
5	0.267	0.559	0.920	1.476	2.015	2.571	4.032	6.869
6	0.265	0.553	0.906	1.440	1.943	2.447	3.707	5.959
7	0.263	0.549	0.896	1.415	1.895	2.365	3.499	5.408
8	0.262	0.546	0.889	1.397	1.860	2.306	3.355	5.041
9	0.261	0.543	0.883	1.383	1.833	2.262	3.250	4.781
10	0.260	0.542	0.879	1.372	1.812	2.228	3.169	4.587
11	0.260	0.540	0.876	1.363	1.796	2.201	3.106	4.437
12	0.259	0.539	0.873	1.356	1.782	2.179	3.055	4.318
13	0.259	0.538	0.870	1.350	1.771	2.160	3.012	4.221
14	0.258	0.537	0.868	1.345	1.761	2.145	2.977	4.140
15	0.258	0.536	0.866	1.341	1.753	2.131	2.947	4.073
16	0.258	0.535	0.865	1.337	1.746	2.120	2.921	4.015
17	0.257	0.534	0.863	1.333	1.740	2.110	2.898	3.965
18	0.257	0.534	0.862	1.330	1.734	2.101	2.878	3.922
19	0.257	0.533	0.861	1.328	1.729	2.093	2.861	3.883
20	0.257	0.533	0.860	1.325	1.725	2.086	2.845	3.850
21	0.257	0.532	0.859	1.323	1.721	2.080	2.831	3.819
22	0.256	0.532	0.858	1.321	1.717	2.074	2.819	3.792
23	0.256	0.532	0.858	1.319	1.714	2.069	2.807	3.768
24	0.256	0.531	0.857	1.318	1.711	2.064	2.797	3.745
25	0.256	0.531	0.856	1.316	1.708	2.060	2.787	3.725
26	0.256	0.531	0.856	1.315	1.706	2.056	2.779	3.707
27	0.256	0.531	0.855	1.314	1.703	2.052	2.771	3.690
28	0.256	0.530	0.855	1.313	1.701	2.048	2.763	3.674
29	0.256	0.530	0.854	1.311	1.699	2.045	2.756	3.659
30	0.256	0.530	0.854	1.310	1.697	2.042	2.750	3.646
60	0.254	0.527	0.848	1.296	1.671	2.000	2.660	3.460
90	0.254	0.526	0.846	1.291	1.662	1.987	2.632	3.402
120	0.254	0.526	0.845	1.289	1.658	1.980	2.617	3.373

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Example 13.2

- Sample: -0.04, -0.19, 0.14, -0.09, -0.14, 0.19, 0.04, and 0.09.
- Mean = 0, Sample standard deviation = 0.138.
- For 90% interval: $t_{[0.95;7]} = 1.895$
- Confidence interval for the mean

$$0 \mp 1.895 \times 0.138 = 0 \mp 0.262 = (-0.262,\ 0.262)$$

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Example 13.5

- Performance: {(5.4, 19.1), (16.6, 3.5), (0.6, 3.4), (1.4, 2.5), (0.6, 3.6), (7.3, 1.7)}. Is one system better?
- Differences: {-13.7, 13.1, -2.8, -1.1, -3.0, 5.6}.

$$\text{Sample mean} = -0.32$$
$$\text{Sample variance} = 81.62$$
$$\text{Sample standard deviation} = 9.03$$
$$\text{Confidence interval for the mean} = -0.32 \mp t\sqrt{(81.62/6)}$$
$$= -0.32 \mp t(3.69)$$
$$t_{[0.95,5]} = 2.015$$
$$90\% \text{ confidence interval} = -0.32 \mp (2.015)(3.69)$$
$$= (-7.75,\ 7.11)$$

- Answer: No. They are not different.

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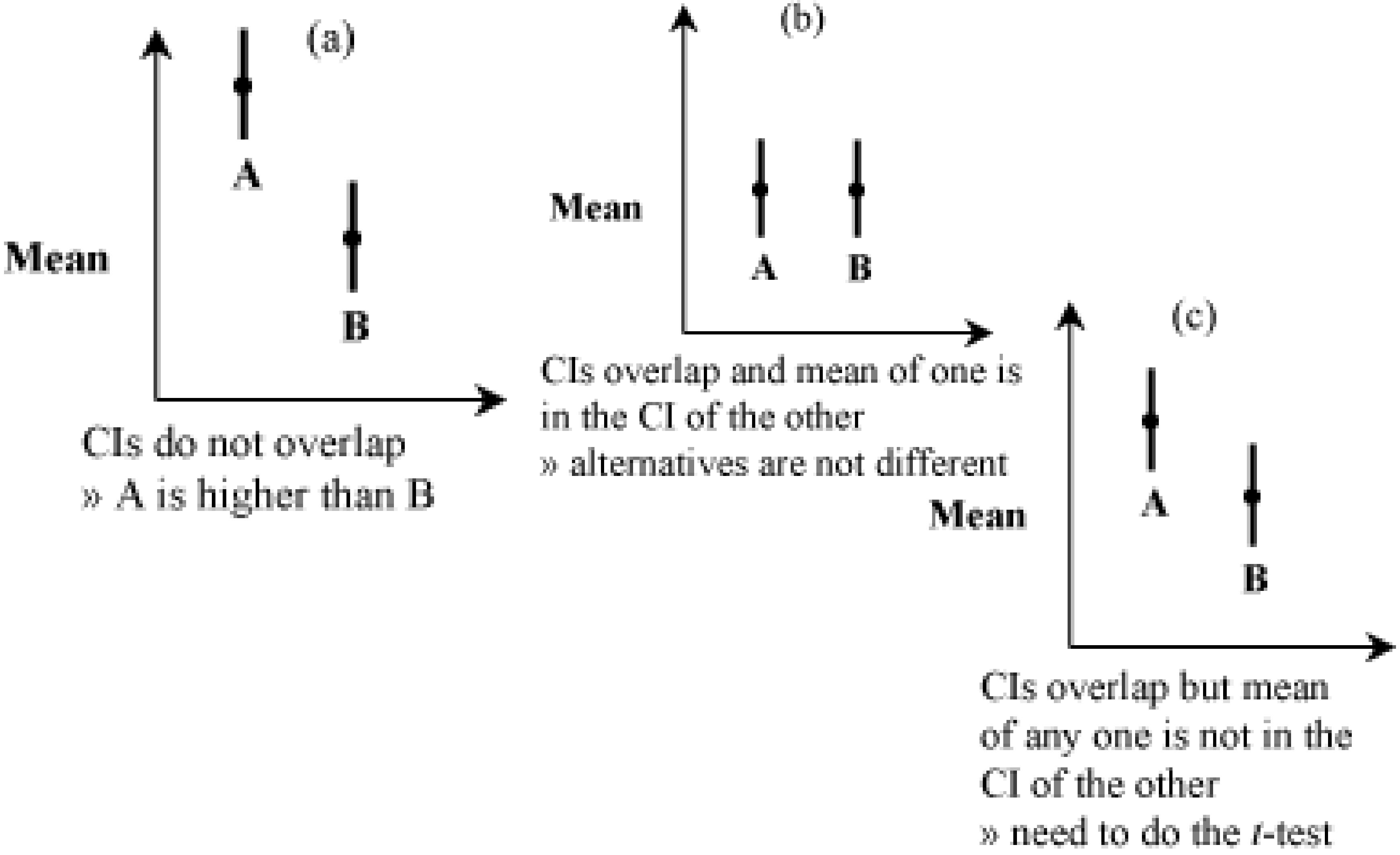
Approximate Visual Test



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Sample Size for Determining Mean

- Larger sample ⇒ Narrower confidence interval ⇒ Higher confidence
- Question: How many observations *n* to get an accuracy of ± r% and a confidence level of 100(1-α)%?
- r% Accuracy ⇒

$$\bar{x} \mp z\frac{s}{\sqrt{n}}$$

⇒

$$\text{CI} = (\bar{x}(1 - r/100), \bar{x}(1 + r/100))$$

$$\bar{x} \mp z\frac{s}{\sqrt{n}} = \bar{x}\left(1 \mp \frac{r}{100}\right)$$

$$z\frac{s}{\sqrt{n}} = \bar{x}\frac{r}{100}$$

$$n = \left(\frac{100zs}{r\bar{x}}\right)^2$$

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Example 13.11

- Sample mean of the response time = 20 seconds

  Sample standard deviation = 5

  Question: How many repetitions are needed to get the response time accurate within 1 second at 95% confidence?
- Required accuracy = 1 in 20 = 5%

  Here, $\bar{x}$= 20, s= 5, z= 1.960, and r=5,

$$n = \left(\frac{(100)(1.960)(5)}{(5)(20)}\right)^2 = (9.8)^2 = 96.04$$

A total of 97 observations are needed.

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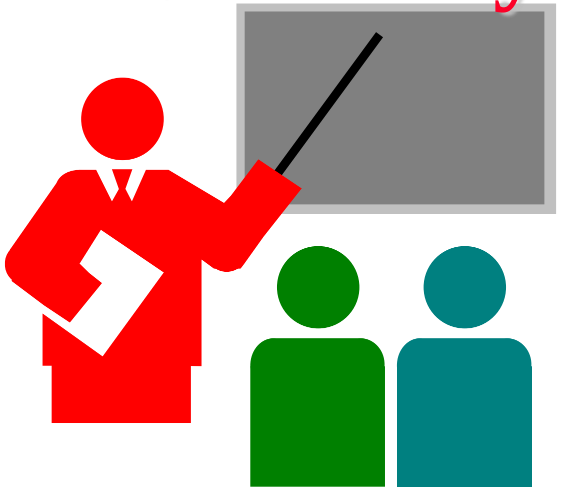
Summary



- All statistics based on a sample are random and should be specified with a confidence interval
- If the confidence interval includes zero, the hypothesis that the population mean is zero cannot be rejected
- Paired observations ⇒ Test the difference for zero mean

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