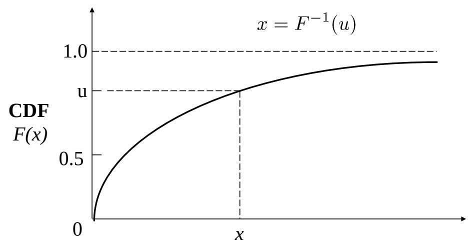
Random Variate Generation

Inverse Transformation

- Used when F^{-1} can be determined either analytically or empirically.

$$u = F(x) \sim U(0,1)$$

$$x = F^{-1}(u)$$



Proof

Let $y = g(x)$, so that $x = g^{-1}(y)$.

$$F_Y(y) = P(Y \le y) = P(x \le g^{-1}(y))$$

$$= F_X(g^{-1}(y))$$

If $g(x) = F(x)$, or $y = F(x)$

$$F(y) = F(F^{-1}(y)) = y$$

And:

$$f(y) = dF/dy = 1$$

That is, y is uniformly distributed between 0 and 1.

Example 28.1

- For exponential variates:

  The pdf $f(x) = \lambda e^{-\lambda x}$
  The CDF $F(x) = 1 - e^{-\lambda x} = u$ or, $x = -\frac{1}{\lambda}\ln(1-u)$

- If u is U(0,1), 1-u is also U(0,1)
- Thus, exponential variables can be generated by:

$$x = -\frac{1}{\lambda}\ln(u)$$

Example 28.2

- The packet sizes (trimodal) probabilities:

Size	Probability
64 Bytes	0.7
128 Bytes	0.1
512 Bytes	0.2

- The CDF for this distribution is:

$$F(x) = \begin{cases} 0.0 & 0 \le x < 64 \\ 0.7 & 64 \le x < 128 \\ 0.8 & 128 \le x < 512 \\ 1.0 & 512 \le x \end{cases}$$

Example 28.2 (Cont)

- The inverse function is:

$$F^{-1}(u) = \begin{cases} 64 & 0 < u \le 0.7 \\ 128 & 0.7 < u \le 0.8 \\ 512 & 0.8 < u \le 1 \end{cases}$$

Generate $u \sim U(0,1)$
$u \le 0.7 \Rightarrow Size = 64$
$0.7 < u \le 0.8 \Rightarrow size = 128$
$0.8 < u \Rightarrow size = 512$

- Note: CDF is *continuous from the right*
  $\Rightarrow$ the value on the right of the discontinuity is used
  $\Rightarrow$ The inverse function is continuous from the left
  $\Rightarrow$ u=0.7 $\Rightarrow$ x=64

Applications of the Inverse-Transformation Technique

Distribution	CDF $F(x)$	Inverse
Exponential	$1-e^{-x/a}$	$-a\ln(u)$
Extreme value	$1-e^{-e^{\frac{x-a}{b}}}$	$a+b\ln\ln u$
Geometric	$1-(1-p)^x$	$\left\lceil\frac{\ln(u)}{\ln(1-p)}\right\rceil$
Logistic	$1-\frac{1}{1+e^{\frac{x-\mu}{b}}}$	$\mu-b\ln(\frac{1}{u}-1)$
Pareto	$1-x^{-a}$	$1/u^{1/a}$
Weibull	$1-e^{(x/a)^b}$	$a(\ln u)^{1/b}$