Testing Random-Number Generators

1. Chi-square test
2. Serial-correlation Test
3. Serial Test

Testing Random-Number Generators

Goal: To ensure that the random number generator produces a random stream.

- Plot histograms
- Plot quantile-quantile plot
- Use other tests
- Passing a test is necessary but not sufficient
- Pass ≠ Good
  Fail ⇒ Bad
- New tests ⇒ Old generators fail the test
- Tests can be adapted for other distributions

Chi-Square Test

- Most commonly used test
- Can be used for any distribution
- Prepare a histogram of the observed data
- Compare observed frequencies with theoretical

k = Number of cells

o_i = Observed frequency for ith cell

e_i = Expected frequency

$$D = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i}$$

- D=0 ⇒ Exact fit
- D has a chi-square distribution with k-1 degrees of freedom.

⇒ Compare D with $\chi^2_{[1-\alpha; k-1]}$ Pass with confidence α if D is less

Example 27.1

- 1000 random numbers with $x_0 = 1$
- $\chi^2_{[0.9;9]} = 14.68$

$$x_n = (125x_{n-1} + 1) \bmod (2^{12})$$

Cell	Obsrvd	Exptd	$\frac{(o-e)^2}{e}$
1	100	100.0	0.000
2	96	100.0	0.160
3	98	100.0	0.040
4	85	100.0	2.250
5	105	100.0	0.250
6	93	100.0	0.490
7	97	100.0	0.090
8	125	100.0	6.250
9	107	100.0	0.490
10	94	100.0	0.360
Total	1000	1000.0	10.380

- Observed difference = 10.380
- Observed is Less ⇒ Accept IID U(0, 1)

Chi-Square for Other Distributions

- Errors in cells with a small e_i affect the chi-square statistic more
- Best when e_i's are equal.

⇒ Use an equi-probable histogram with variable cell sizes

- Combine adjoining cells so that the new cell probabilities are approximately equal.
- The number of degrees of freedom should be reduced to k-r-1 (in place of k-1), where r is the number of parameters estimated from the sample.
- Designed for discrete distributions and for large sample sizes only ⇒ Lower significance for finite sample sizes and continuous distributions
- If less than 5 observations, combine neighboring cells

Serial-Correlation Test

- Nonzero covariance ⇒ Dependence. The inverse is not true
- R_k = Autocovariance at lag k = Cov[x_n, x_{n+k}]

$$R_k = \frac{1}{n-k}\sum_{i=1}^{n-k}(U_i - \frac{1}{2})(U_{i+k} - \frac{1}{2})$$

- For large n, R_k is normally distributed with a mean of zero and a variance of 1/[144(n-k)]
- 100(1-α)% confidence interval for the autocovariance is:

$$R_k \mp z_{1-\alpha/2}/(12\sqrt{n-k})$$

For k¸ 1 Check if CI includes zero

- For k = 0, R_0= variance of the sequence Expected to be 1/12 for IID U(0,1)

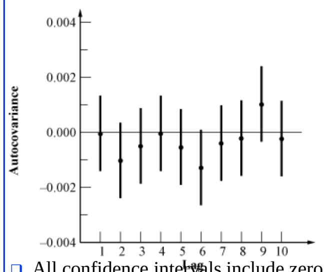
Example 27.3: Serial Correlation Test

$$x_n = 7^5 x_{n-1} \bmod (2^{31}-1)$$

10,000 random numbers with x_0=1:

Lag k	Autocovariance R_k	St. Dev. of R_k	90% Confidence Interval Lower Limit	Upper Limit
1	-0.000038	0.000833	-0.001409	0.001333
2	-0.001017	0.000833	-0.002388	0.000354
3	-0.000489	0.000833	-0.001860	0.000882
4	-0.000033	0.000834	-0.001404	0.001339
5	-0.000531	0.000834	-0.001902	0.000840
6	-0.001277	0.000834	-0.002648	0.000095
7	-0.000385	0.000834	-0.001757	0.000986
8	-0.000207	0.000834	-0.001579	0.001164
9	0.001031	0.000834	-0.000340	0.002403
10	-0.000224	0.000834	-0.001595	0.001148

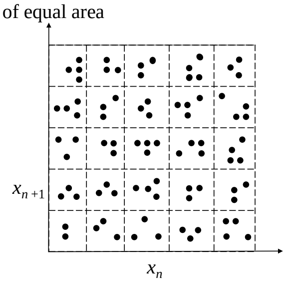
Example 27.3 (Cont)



- All confidence intervals include zero ⇒ All covariances are statistically insignificant at 90% confidence.

Serial Test

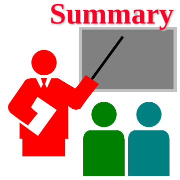
- Goal: To test for uniformity in two dimensions or higher.
- In two dimensions, divide the space between 0 and 1 into K^2 cells of equal area



Serial Test (Cont)

- Given {$x_1, x_2, \ldots, x_n$}, use n/2 non-overlapping pairs (x_1, x_2), (x_3, x_4), … and count the points in each of the K^2 cells.
- Expected= n/(2K^2) points in each cell.
- Use chi-square test to find the deviation of the actual counts from the expected counts.
- The degrees of freedom in this case are K^2-1.
- For k-dimensions: use k-tuples of non-overlapping values.
- k-tuples must be non-overlapping.
- Overlapping ⇒ number of points in the cells are not independent chi-square test cannot be used
- In visual check one can use overlapping or non-overlapping.
- In the spectral test overlapping tuples are used.
- Given n numbers, there are n-1 overlapping pairs, n/2 non-overlapping pairs.

Summary



1. Chi-square test is a one-dimensional test
   Designed for discrete distributions and large sample sizes
2. Serial correlation test for independence
3. k-dimensional uniformity = k-distributivity
   tested by spectral test or serial test